

Distance-residual graphs

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Abstract

If we are given a connected finite graph G and a subset of its vertices V_0 , we define a distance-residual graph as a graph induced on the set of vertices that have the maximal distance from V_0 . Some properties and examples of distance-residual graphs of vertex-transitive, edge-transitive, bipartite and semisymmetric graphs are shown. The relations between the distance-residual graphs of product graphs and their factors are shown.

Keywords: distance-residual graphs, product graphs, vertex-transitive graphs, edge-transitive graphs, semisymmetric graphs, Gray graph

Mathematics Subject Classification: 05C12

1 Introduction

Let G be a connected finite graph and let $V_0 \subset V(G)$ be a nonempty subset of vertices of G . We may form a distance partition $P(G, V_0)$ of G with respect to V_0 , where $P(G, V_0) = \{V_0, V_1, \dots, V_r\}$ and

$$\begin{aligned} V_0 \cup V_1 \cup \dots \cup V_r &= V(G), \\ V_i \cap V_j &= \emptyset, \text{ for } i \neq j, \\ V_i &\neq \emptyset, \text{ for all } i. \end{aligned}$$

The classes V_i are defined recursively as

$$V_{i+1} := \{v \in V(G) \setminus (V_0 \cup V_1 \cup \dots \cup V_i) \mid \exists u \in V_i : [u, v] \in E(G)\}.$$

The set V_i contains all vertices of G that have a minimal distance of i to the vertices of V_0 where the distance $d(u, v)$ between two vertices u and v is defined as the shortest path between them. Therefore $d(v, v_i) \geq i$ for $v \in V_0$ and $v_i \in V_i$, and there exists a vertex $v_0 \in V_0$ for which $d(v_0, v_i) = i$.

We are interested in induced subgraphs $\langle V_i \rangle$ defined by the distance classes, particularly in the subgraph with the lowest index $R_G := \langle V_0 \rangle$, which we call the *root*, and the subgraph with highest index $Res(G, R_G) := \langle V_r \rangle$, which we call the *distance-residual graph* or the *distance residual*. When the root consists of a single vertex, i.e. $R_G \cong K_1$, the residual is called a *vertex residual*. When $R_G \cong K_2$, the residual is called an *edge residual*. With respect to the definition of the distance residuals, all of the graphs in the paper will be nontrivial, simple, finite, and in most cases connected. Also some standard labels for some known graphs will be used: K_n for complete graphs, $K_{m,n}$ for complete bipartite graphs, C_n for cycles, P_n for paths, and mK_n for a disjoint union of m complete graphs on n vertices.

The motivation for the definition of distance-residual graphs was in extending the definition of *distance sequence* which is an ordered list where the i -th element equals the number of vertices at distance i from the selected root. The distance sequence therefore presents only the number of vertices at a distance i from the root but we are also interested in the induced subgraphs on those vertices, especially on the set farthest away from the root.

In the next section we present some properties of the distance-residual graphs with the focus on the vertex- and edge-transitive graphs [1, 2], bipartite graphs, and semisymmetric graphs [8, 21]. In section 3 we show how the distance residuals of product graphs for some well-known products depend on the distance residuals of their factors. We conclude with some open questions regarding distance residuals and other distance related problems.

2 Properties

We can only define a distance-residual graph of a connected graph but any graph, connected or not, can be a distance-residual graph.

Theorem 2.1 *Let H be an arbitrary graph and $n \in \mathbb{N}$. Then there exists a connected graph G with the root R_G of order n such that H is isomorphic to $Res(G, R_G)$.*

Proof: Let us choose an arbitrary graph R_G of order n with the property $V(R_G) \cap V(H) = \emptyset$. Graph G is constructed as follows: The vertex set $V(G)$ consists of $V(H) \cup V(R_G)$. There are two types of edges in G . All the original edges of H and R_G remain edges in G and for each vertex $v \in V(H)$ and each vertex $r \in V(R_G)$ there is an edge between them. Clearly, the distance partition is given by $V_0 = V(R_G)$ and $V_1 = V(H)$, therefore $Res(G, R_G) = H$. \square

Most of the interesting cases occur for vertex-transitive and edge-transitive graphs.

2.1 Vertex- and edge-transitive graphs

Lemma 2.1 *Let G be a connected vertex-transitive graph. Then all of its vertex residuals are isomorphic, i.e. they do not depend on the choice of the vertex for the root. The converse is not true even if the graph is regular.*

Proof: The first part is obvious because if the vertex residuals would not be isomorphic, the graph would not have a transitive automorphism group. The graph in Figure 1 proves that the converse is not true. It is built from two copies of K_4 by joining some of their edges (see [16] for a definition of joining) and is therefore 3-regular. All of its vertex residuals are isomorphic (to K_1) but it is not vertex-transitive because the automorphism, which would map a vertex from K_4 to a vertex created by joining edges, does not exist.

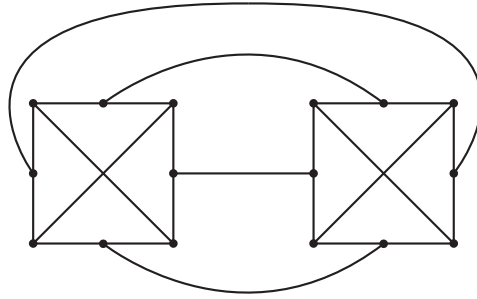


Figure 1: Non-vertex-transitive cubic graph with isomorphic vertex residuals.

□

Vertex residual is also connected to the *growth* of the graph (see [30]) because its order is the leading coefficient of the growth polynomial of the graph at its root. This is equivalent to taking the last element of the distance sequence of the root.

We know that there exist graphs that are *growth-regular*, i.e. their growth function is independent of the root vertex, but not vertex-transitive. From the above proof we can also see that such graphs can have isomorphic vertex residuals and still not be vertex-transitive. The growth function of the graph in Figure 1 is namely $1 + 3x + 6x^2 + 5x^3 + x^4$ independent of the vertex we take for the root.

Lemma 2.2 *Let G be a connected edge-transitive graph. Then all of its edge residuals are isomorphic, i.e. they do not depend on the choice of the two adjacent vertices for the root. The converse is not true.*

Proof: The first part is similar as in the prior lemma. As a counterexample of the converse we present the graph in Figure 2 that has all of its edge residuals isomorphic (to K_1) but it is clearly not edge-transitive.

□

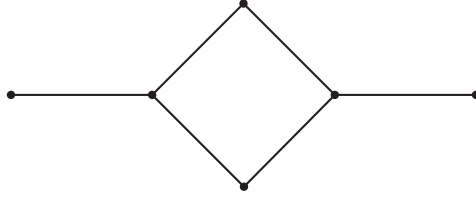


Figure 2: Non-edge-transitive graph with isomorphic edge residuals.

Lemma 2.3 *Let G be a connected edge-transitive graph and $L(G)$ its line graph. Then all of the distance residuals $Res(L(G), K_1)$ are isomorphic.*

Proof: The line graph of a connected edge-transitive graph is a connected vertex-transitive graph. The rest follows from Lemma 2.1. \square

We present the distance residuals of some well-known graphs.

Example 2.1

$$\begin{array}{ll}
 Res(C_{2k}, K_1) \cong K_1 & Res(C_{2k}, K_2) \cong K_2 \\
 Res(C_{2k+1}, K_1) \cong K_2 & Res(C_{2k+1}, K_2) \cong K_1 \\
 Res(K_n, K_1) \cong K_{n-1} & Res(K_n, K_k) \cong K_{n-k}
 \end{array}$$

Example 2.2 *The Petersen graph $P(5, 2)$ is vertex-transitive, edge-transitive and also distance-transitive, therefore it has isomorphic distance residuals for some roots:*

$$\begin{array}{l}
 Res(P(5, 2), K_1) \cong C_6, \\
 Res(P(5, 2), K_2) \cong 2K_2, \\
 Res(P(5, 2), P_3) \cong 2K_1, \\
 Res(P(5, 2), P_4) \cong K_1, \\
 Res(P(5, 2), C_5) \cong C_5.
 \end{array}$$

2.2 Bipartite graphs

For bipartite graphs a special relation between vertex and edge residuals holds. But before we present it, we shall define the distance between vertices and subgraphs. Let $d(R, v)$ be the distance from $R \subset G$ to vertex $v \in V(G)$, i.e. $d(R, v) := \min_{r \in V(R)} d(r, v)$, and let $d(R, H) := \min_{v \in V(H)} d(R, v)$ be the distance between two subgraphs of graph G . If graph G is connected, the distances are well defined.

We will often need the distance between the root and the distance residual of some connected graph G , so we will denote $d(R_G, Res(G, R_G))$ as d_{R_G} . The distance namely depends on the root of the graph.

In the next theorem we use the above notation to shorten the writing for the distance $d(\langle\{u\}\rangle, \text{Res}(G, \langle\{u\}\rangle))$ to d_u and similarly to shorten $d(\langle\{u, v\}\rangle, \text{Res}(G, \langle\{u, v\}\rangle))$ to $d_{u,v}$.

Theorem 2.2 *Let $G \not\cong K_2$ be a connected bipartite graph with partitions P_1 and P_2 . Let $u \in P_1$ and $v \in P_2$ be two neighbors from G . Then the edge residual equals*

$$\text{Res}(G, \langle\{u, v\}\rangle) = \begin{cases} \langle V(\text{Res}(G, \langle\{u\}\rangle)) \cup V(\text{Res}(G, \langle\{v\}\rangle)) \rangle, & \text{if } d_u = d_v; \\ \text{Res}(G, \langle\{u\}\rangle), & \text{if } d_u = d_v + 1; \\ \text{Res}(G, \langle\{v\}\rangle), & \text{if } d_v = d_u + 1. \end{cases}$$

There are no other possibilities.

Proof: First we prove that the three cases above are the only ones possible. Let us assume that $d_u > d_v + 1$ and take a vertex $r \in V(\text{Res}(G, \langle\{u\}\rangle))$. From $d_v \geq d(v, r)$ we get $d_u > d(v, r) + 1$. But that means there is a shorter path between u and r via v (u and v are neighbors) which is a contradiction. The proof is similar if we assume $d_v > d_u + 1$. From this we can also deduce that $d(u, s) = d(v, s) \pm 1$ for any vertex $s \in V(G)$; because the graph is bipartite, the distances cannot be equal.

If d_u and d_v are equal, the vertex residuals lie in different partitions. If we take an arbitrary vertex $r \in V(\text{Res}(G, \langle\{u\}\rangle))$, then $d(v, r) = d_u - 1$ (if $d(v, r) = d_u + 1$, then $d_v > d_u$). The same is true if we change the role of u and v . So $d_{u,v} \geq d_u - 1 = d_v - 1$.

To prove that $d_{u,v}$ is equal to $d_u - 1$ and that the edge-residual graph is induced only on the vertices of both vertex residuals we take a vertex s which is not in any of the vertex residuals. If $d_u > d(u, s) + 1$, then $d_u - 1 > d(u, s)$. Let $d_u = d(u, s) + 1$. Then $d(v, s) = d(u, s) - 1$ because if $d(v, s) = d(u, s) + 1 = d_u = d_v$, s would be in the vertex residual of v . It follows that $d(\langle\{u, v\}\rangle, s) = d(u, s) - 1 = d_u - 2 < d_u - 1$. The proof is the same if we interchange u and v .

If $d_u = d_v + 1$ and we take a vertex $r \in V(\text{Res}(G, \langle\{u\}\rangle))$, then $d_u = d(u, r) = d(v, r) + 1$, therefore $d(v, r) = d_v$ and $r \in V(\text{Res}(G, \langle\{v\}\rangle))$. It follows that $d_{u,v} = d_v$. If we take a vertex s from the vertex residual of v but not of u , then $d_v = d(v, s) = d(u, s) + 1$. Therefore $d(\langle\{u, v\}\rangle, s) = d(u, s) = d(v, s) - 1 < d_v$ and s is not in the edge residual. The proof in the third case is essentially the same; we just interchange u and v .

□

Example 2.3 *Let $K_{m,n}$ ($m, n \neq 1$) have partitions sets $|P_1| = m$ and $|P_2| = n$. Let $u \in P_1$ and $v \in P_2$. Then $\text{Res}(K_{m,n}, \langle\{u\}\rangle) \cong (m-1)K_1$ and $\text{Res}(K_{m,n}, \langle\{v\}\rangle) \cong (n-1)K_1$. Because $d_u = d_v = 2$, we get from Theorem 2.2 $\text{Res}(K_{m,n}, K_2) \cong (m+n-2)K_1$.*

An interesting class of bipartite graphs are *semisymmetric graphs*, i.e. regular graphs which are edge-transitive but not vertex-transitive. Semisymmetric graphs have an automorphism group that acts transitively on each of the bipartition sets. That means

the vertex residuals of roots from the same set are isomorphic. Furthermore, the distance sequences in each of the sets are also the same.

In fact, one motivation for the definition of distance-residual graphs was the fact that the *Gray graph* [19, 20], the smallest semisymmetric cubic graph [11], is the edge residual of the generalized quadrangle $W(3)$ [15, 24] as mentioned in [28]. Gray graph is of order 54 and has the distance sequences $(1, 3, 6, 12, 12, 12, 8)$ and $(1, 3, 6, 12, 16, 12, 4)$ with the vertex residuals $8K_1$ and $4K_1$ depending on the partition from which the root vertex was taken. The edge residual is induced on the 12 vertices of both vertex residuals and is isomorphic to $4P_3$.

We mention in passing that there is an error in [19] in the Figure 7 which represents the construction 2.4 of the Gray graph. The vertex labels a_i and b_i (for $1 \leq i \leq 4$) in the auxiliary graph G_{2345} must be interchanged.

The smallest semisymmetric graph is the *Folkman graph* of order 20 and valence 4 which has distance sequences $(1, 4, 9, 6)$ and $(1, 4, 6, 6, 3)$, and vertex residuals $6K_1$ and $3K_1$. By Theorem 2.2 the edge residual graph equals $3K_1$.

For another example we look at the so called *Ljubljana graph* [10] of order 112 with the distance sequences $(1, 3, 6, 12, 24, 34, 24, 7, 1)$ and $(1, 3, 6, 12, 24, 34, 25, 7)$. The vertex residuals are isomorphic to K_1 and $7K_1$ with the edge residual equal to K_1 . The Ljubljana graph was originally discovered by R. Foster (unpublished) and later studied in a series of papers by I.J. Dejter and his co-authors [7, 9, 13, 14]. Only in [10], where the Ljubljana graph was rediscovered for the third time, it was determined that it is the unique third smallest cubic semisymmetric graph, and hence isomorphic to the graph of Foster, Dejter, et al.

3 Distance residuals of product graphs

Product graphs have various interesting properties that make them subject of intensive studies. Problems, that are intractable for general graphs, sometimes admit elegant solutions for special classes of graphs, such as product graphs. This fact frequently drew our attention in the past [4, 6, 12, 23, 25, 26, 27].

In this section we show how distance-residual graphs of product graphs depend on the factors of those products and on their respective distance-residual graphs. All of the well-known graph products are covered: Cartesian, strong, direct, and lexicographic product (see [17] for more about graph products and their properties). This simplifies the discovery of distance-residual graphs in some well-known graphs as well as proving some interesting properties of vertex-transitive graphs.

3.1 Cartesian product

We start with the Cartesian product (denoted as $G \square H$), which is the most fundamental and the most studied of all. Its vertex set is, like the sets of all the other here mentioned products, defined on the Cartesian product $V(G) \times V(H)$ of the vertex sets of the factors.

Its edge set is the set of all pairs $[(u, x), (v, y)]$ where either $u = v$ and $[x, y] \in E(H)$ or $x = y$ and $[u, v] \in E(G)$.

The distance-residual graph of the Cartesian product of two graphs can easily be expressed with the distance-residual graphs of the respective factors.

Theorem 3.1 *Let G and H be two connected graphs with $R_G \subset G$ and $R_H \subset H$ as their roots. We can state the following connection between distance-residual graphs:*

$$Res(G \square H, R_G \square R_H) \cong Res(G, R_G) \square Res(H, R_H).$$

Proof: Because both factors are connected, so is the product graph and therefore the distance-residual graph $Res(G \square H, R_G \square R_H)$ is well defined. The distances in the product are sums of distances in both factors [17], i.e. the distance between two vertices (g_1, h_1) and (g_2, h_2) in $G \square H$ is equal to $d_G(g_1, g_2) + d_H(h_1, h_2)$. Therefore

$$d_{G \square H}(R_G \square R_H, (g, h)) = d_G(R_G, g) + d_H(R_H, h),$$

because

$$\begin{aligned} \min_{(r_G, r_H) \in V(R_G \square R_H)} d_{G \square H}((r_G, r_H), (g, h)) &= \min_{(r_G, r_H) \in V(R_G \square R_H)} d_G(r_G, g) + d_H(r_H, h) \\ &= \min_{r_G \in V(R_G)} d_G(r_G, g) + \min_{r_H \in V(R_H)} d_H(r_H, h) \\ &= d(R_G, g) + d(R_H, h). \end{aligned}$$

So the residual graph contains the vertex (g, h) if and only if $g \in V(Res(G, R_G))$ and $h \in V(Res(H, R_H))$. And finally, because the distance-residual graph is induced, we must take the Cartesian product of distance-residual graphs of both factors. \square

From the associativity of the Cartesian product it also follows:

Corollary 3.1

$$Res(G_1 \square \cdots \square G_n, R_{G_1} \square \cdots \square R_{G_n}) \cong Res(G_1, R_{G_1}) \square \cdots \square Res(G_n, R_{G_n}).$$

Example 3.1 *The n -dimensional hypercube Q_n is the Cartesian product of n copies of K_2 , so its vertex residual is K_1 . Also, if some graph in the product has a trivial vertex residual, we do not have to take it into consideration.*

3.2 Strong product

Strong product, which is denoted by $G \boxtimes H$, is similar to the Cartesian product because the edge set of the product includes the same edges as the Cartesian product with the addition of those edges $[(u, x), (v, y)]$ where $[u, v] \in E(G)$ and $[x, y] \in E(H)$.

There are three possible options for the distance-residual graph of the strong product, depending on the distances from roots to the points in distance-residual graphs in both factors. So we reuse the notation d_{R_G} as the distance $d(R_G, \text{Res}(G, R_G))$ from the root to the distance-residual graph of some graph G .

Theorem 3.2 *Let G and H be two connected graphs with $R_G \subset G$ and $R_H \subset H$ as their roots. The distance-residual graph of the strong product of these graphs is*

$$\text{Res}(G \boxtimes H, R_G \boxtimes R_H) \cong \begin{cases} H \boxtimes \text{Res}(G, R_G), & \text{if } d_{R_G} > d_{R_H}; \\ G \boxtimes \text{Res}(H, R_H), & \text{if } d_{R_G} < d_{R_H}; \\ \langle V(\text{Res}(G, R_G)) \times V(H) \cup \\ V(G) \times V(\text{Res}(H, R_H)) \rangle, & \text{else.} \end{cases}$$

Proof: We follow the proof presented in the case of the Cartesian product. Here also the graph of the strong product is connected so the distance-residual graph is well defined. The distance in the product is equal to the maximal distance in both factors [17], therefore

$$d_{G \boxtimes H}(R_G \boxtimes R_H, (g, h)) = \max\{d_G(R_G, g), d_H(R_H, h)\}.$$

If $d_{R_G} > d_{R_H}$, it follows $d_{G \boxtimes H}(R_G \boxtimes R_H, (g, h)) = d(R_G, g)$. So the distance-residual graph contains the vertex (g, h) if and only if $g \in V(\text{Res}(G, R_G))$. Vertex h is therefore an arbitrary vertex from graph H and because the distance-residual graph is induced, we get the mentioned result. The same argument follows when $d_{R_G} < d_{R_H}$.

In the case where both distances are equal we get a distance-residual graph which comprises of all the vertices of $\text{Res}(G, R_G) \times H$ and $G \times \text{Res}(H, R_H)$ (with the vertices of $\text{Res}(G, R_G) \times \text{Res}(H, R_H)$ represented twice). Once again, because the distance-residual graph is induced, we get the mentioned result. \square

Let \mathcal{G}_{\max} be the set of all connected graphs G_1, \dots, G_n with their respective roots R_{G_1}, \dots, R_{G_n} for which the distance $d_{R_{G_i}}$ is maximal. To put it in another way, $\mathcal{G}_{\max} := \{G_j \mid d_{R_{G_j}} = \max_{1 \leq i \leq n} \{d_{R_{G_i}}\}\}$. Let us rearrange the graphs so that the members of \mathcal{G}_{\max} get the indexes from one to $k \leq n$. In a similar way as with the Cartesian product we can come to the following conclusion:

Corollary 3.2 *If \mathcal{G}_{\max} has k ($1 \leq k \leq n$) elements, then the distance-residual graph $\text{Res}(G_1 \boxtimes \dots \boxtimes G_n, R_{G_1} \boxtimes \dots \boxtimes R_{G_n})$ is isomorphic to*

$$\left\langle \bigcup_{1 \leq i \leq k} V(G_1) \times \dots \times V(G_{i-1}) \times V(\text{Res}(G_i, R_{G_i})) \times V(G_{i+1}) \times \dots \times V(G_n) \right\rangle. \quad (1)$$

If \mathcal{G}_{\max} has only one member (G_1), the graph in (1) can be written as

$$\text{Res}(G_1, R_{G_1}) \boxtimes G_2 \boxtimes \dots \boxtimes G_n.$$

Example 3.2 *The complete graph on mn vertices is the strong product of K_m with K_n . Therefore, $\text{Res}(K_{mn}, K_s \boxtimes K_t) \cong K_{mn-st}$ for $s < m$ and $t < n$.*

3.3 Lexicographic product

The major difference between the lexicographic product (denoted here as $G \circ H$) and the products mentioned before is that this product is not commutative. The lack of this symmetry can be seen from the definition of edges because $[(u, x), (v, y)] \in E(G \circ H)$ if either $[u, v] \in E(G)$ or $u = v$ and $[x, y] \in E(H)$.

The product is sometimes written as $G[H]$ and can be represented by replacing each vertex $v \in V(G)$ with the copy of graph H , denoted H_v , and then connecting all off the vertices of H_u with all off the vertices of H_v if the vertices u and v are neighbors in G . The distance between vertices in the product therefore depends on whether they lie in the same copy of H . If they do, they are at a distance two (or one if they are neighbors), if they do not, their distance is determined by the distance of their first factors, i.e. the distance between the copies of H in which they lie in. Therefore $d_{R_{G \circ H}}$ can be equal to 1, 2 or d_{R_G} .

From the above description we can see that if G is connected (and nontrivial) then so is $G \circ H$ for arbitrary H . The statement is true also in the other direction (see [17]).

The same as with the strong product, the distance-residual graph depends on the distances between roots and distance-residual graphs of its factors. Therefore, we use the notation established there.

Theorem 3.3 *Let G and H be two graphs with $R_G \subset G$ and $R_H \subset H$ as their roots and let G be connected. The distance-residual graph of the lexicographic product of these graphs, $Res(G \circ H, R_G \circ R_H)$, is isomorphic to*

1. $\bigcup_{\substack{r_G \in V(R_G) \\ r_G \text{ isolated}}} (H \setminus \langle V_0 \cup V_1 \rangle_H),$ if $d_{R_G} = 1$, $d_{R_H} \neq 1$ and there exist isolated vertices in R_G ,
2. $(G \circ H) \setminus (R_G \circ R_H),$ if $d_{R_G} = 1$ and ($d_{R_H} = 1$ or R_G has no isolated vertices),
3. $Res(G, R_G) \circ H \cup \bigcup_{\substack{r_G \in V(R_G) \\ r_G \text{ isolated}}} (H \setminus \langle V_0 \cup V_1 \rangle_H),$ if $d_{R_G} = 2$,
4. $Res(G, R_G) \circ H,$ if $d_{R_G} \geq 3$.

Proof:

1. If $d_{R_G} = 1$, then all of the vertices in H_v , where $v \notin V(R_G)$, lie in the class V_1 of the distance partition of $G \circ H$. If two vertices u and v from R_G are adjacent, then the vertices of $V(H_v) \setminus V(R_{H_v})$ and $V(H_u) \setminus V(R_{H_u})$ also lie in V_1 . But if v is an isolated vertex of R_H , i.e. its neighbors are from $V(G) \setminus V(R_G)$, then the vertices of $H_v \setminus \langle V_0 \cup V_1 \rangle_{H_v}$ (which exist because $d_H \neq 1$) lie in V_2 and therefore induce the distance-residual graph of the product.

2. If $d_{R_G} = 1$ and R_G has no isolated vertices, then, following the above proof, we see that all of the vertices, which are not from $R_G \circ R_H$, lie in V_1 . The same follows if $d_{R_H} = 1$.
3. If $d_{R_G} = 2$, then for all $v \in V(\text{Res}(G, R_G))$ the vertices of H_v lie in V_2 of the partition $P(G \circ H, V_0)$. But from the proof of the first case we see that V_2 also consists of vertices from $H_v \setminus \langle V_0 \cup V_1 \rangle_H$ for isolated $v \in V(R_G)$. If there are no isolated vertices in R_G or if $d_{R_H} = 1$, then the latter vertices do not exist.
4. This follows from the third case because for all $v \in V(\text{Res}(G, R_G))$ the vertices of H_v lie in $V_{d_{R_G}}$ of the partition $P(G \circ H, V_0)$ which induces the distance-residual graph.

□

We can generalize the result in case 4 of the Theorem 3.3 to more factors with the help of the associativity of the lexicographic product and calculate it recursively.

Corollary 3.3 *Let G_1, G_2, \dots, G_n be graphs with their respective roots $R_{G_1}, R_{G_2}, \dots, R_{G_n}$ and let G_1 be connected. Also, let the distance $d_{R_{G_1} \circ \dots \circ R_{G_i}}$ in $G_1 \circ \dots \circ G_i$ be at least 3 for all $i \in \{1, \dots, n-1\}$. It follows that*

$$\text{Res}(G_1 \circ G_2 \circ \dots \circ G_n, R_{G_1} \circ R_{G_2} \circ \dots \circ R_{G_n}) \cong \text{Res}(G_1, R_{G_1}) \circ G_2 \circ \dots \circ G_n.$$

With the use of the lexicographic product we can prove another property of the distance-residual graphs in vertex-transitive graphs.

Theorem 3.4 *Let H be a vertex-transitive graph and $n \in \mathbb{N}$. Then there exists a connected vertex-transitive graph G with the root R_G of order n such that H is isomorphic to $\text{Res}(G, R_G)$.*

Proof: For the construction of graph G we use the lexicographic product of graphs C_k and H . Because both of them are vertex-transitive so is their product G (see [17]) and because C_k is connected so is the product. Let $R_H := K_1$ and $R_{C_k} := P_n$. Therefore $|R_H \circ R_{C_k}| = n$. If we take $k = n + 5$, then $d_{R_{C_k}} = 3$. The $\text{Res}(C_{n+5} \circ H, R_{C_{n+5}} \circ R_H)$ is by case 4 of Theorem 3.3 isomorphic to $\text{Res}(C_{n+5}, P_n) \circ H \cong K_1 \circ H \cong H$. □

Example 3.3 *With the help of Theorem 3.4 we can construct a vertex-transitive graph that has all of its vertex residuals isomorphic to the Petersen graph. But the latter is also the vertex residual of the Clebsch graph shown on Figure 3, which is a symmetric (vertex- and edge-transitive), strongly regular graph on 16 vertices of valence 5.*

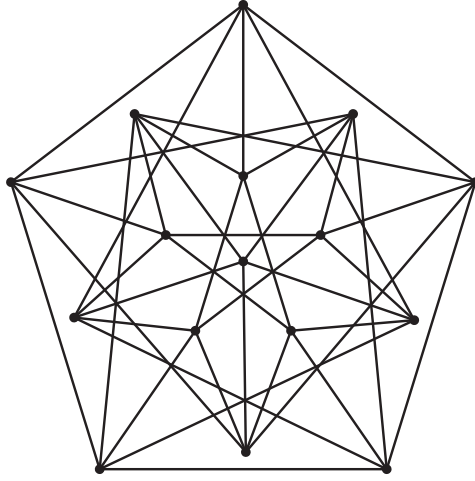


Figure 3: The Clebsch graph which has all of its vertex residuals isomorphic to the Petersen graph.

3.4 Direct product

Finally, we mention the direct product which is also known as tensor product or categorical product [17]. We denote it by $G \times H$ (the sign \otimes is also frequently used). Its edge set is made up of edges $[(u, x), (v, y)]$ where $[u, v] \in E(G)$ and $[x, y] \in E(H)$.

The product has some unusual properties. First of all, the connectivity of both factors is not sufficient condition for the connectivity of the product; at least one of them must not be bipartite. Furthermore, the distance function between vertices in the product is unlike with the other products, as shown by the following lemma.

Lemma 3.1 (see [5]) *Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be two vertices of $G := G_1 \times G_2 \times \dots \times G_n$. If there is no integer m for which each G_i has an $x_i - y_i$ walk of length m , then $d_G(x, y) = \infty$. Otherwise, $d_G(x, y) = \min\{m \in \mathbb{N} \mid \text{each } G_i \text{ has an } x_i - y_i \text{ walk of length } m\}$.*

We present a theorem that follows from Lemma 3.1 from which we can conclude that the distance-residual graph of the direct product does not depend on the residuals of the factors.

Theorem 3.5 *Let G and H be connected graphs and at least one of them non-bipartite. Let R_G and R_H be their respective roots. Then the distance-residual graph $\text{Res}(G \times H, R_G \times R_H)$ is induced on all of the vertices $(u, v) \in V(G \times H)$ for which the following holds:*

$$d(R_G \times R_H, (u, v)) = \max_{(g, h) \in V(G \times H)} \min_{r_G \in R_G, r_H \in R_H} \{m \in \mathbb{N} \mid G \text{ has an } r_G - g \text{ walk of length } m \text{ and } H \text{ has an } r_H - h \text{ walk of length } m\}.$$

Example 3.4 Let $G := P_3$ with R_G consisting of one of the two noncentral vertices of P_3 . Let H be a graph consisting of K_3 with an appended edge on each of the vertices. For the root of H we take one of the vertices not from K_3 . Then $\text{Res}(P_3 \times H, R_G \times R_H) \cong K_1$ with the vertex having as the first component the central vertex of P_3 and the second from the root of H .

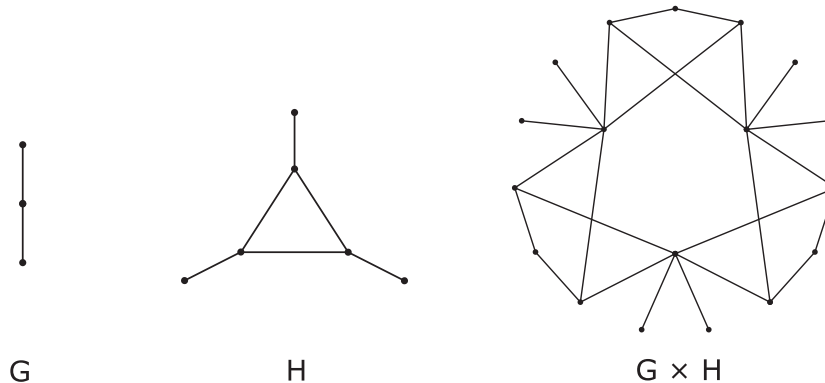


Figure 4: Graphs G and H from Example 3.4 and their direct product

4 Concluding remarks

In this paper we have defined a distance-residual graph and proven some of its properties. It would be interesting to see whether our methods could be used to answer some of the following questions.

Since every graph can be a distance-residual graph, it is a challenge to find well-known graphs as distance residuals of some other well-known graphs. We also ask what is the sufficient condition for a growth-regular graph, i.e. a graph with the same distance sequences, to be vertex-transitive. Graph bundles [3, 18, 22, 29, 31] form an interesting generalizations of product graphs. It would be of interest to investigate their properties in connection to residual graphs. And finally, regular edge-transitive graph that admits two distinct distance sequences is necessarily semi-symmetric. In principle, the converse need not be true. It would be interesting to apply our methods for construction of families of semi-symmetric graphs in which all vertices give rise to the same distance sequence.

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